Indian Statistical Institute, Bangalore
B. Math. Second Year
Second Semester - Computer Science II

Final Exam | Duration: 3 hours | Date : Apr 26, 2017

Answer all the questions. Max Marks: 50

1. Define a matrix norm and explain how it differs from a vector norm . Show that the vector norm
$$\|x\|_\infty = \max_i |x_i|$$
for $x \in \mathbb{R}^n$ induces the matrix norm
$$\|\boldsymbol{A}\|_\infty = \max_i \sum_j |a_{ij}|.$$
[**10**]

2. Let $\boldsymbol{A}$ be an $n \times n$ nonsingular matrix with real entries. If $\|\boldsymbol{A}\|_\infty < 1$ show that $(\boldsymbol{I} - \boldsymbol{A})^{-1}$ exists and $(\boldsymbol{A} + \boldsymbol{E})^{-1}$ also exists provided $\|\boldsymbol{A^{-1}E}\|_\infty < 1$. Now consider the matrix $\boldsymbol{B}$ with real entries
$$\begin{bmatrix} \epsilon & 0 & 0 & a \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ b & 0 & 0 & \epsilon \end{bmatrix}$$
and establish conditions on a, b for $\boldsymbol{B}^{-1}$ to exist for $\epsilon \neq 0$.
[**10**]

3. State and prove the Gerschgorin circle theorem for a real $n \times n$ matrix A. If A is a strictly diagonal matrix with nonzero diagonal entries show that it is invertible. Using this theorem discuss the location of eigenvalues for the matrix
$$\begin{bmatrix} 2 & -1 & 0 \\ 1 & 4 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$
[**10**]

4. Let $x_0, \cdots, x_n \in [a, b]$, $f : [a, b] \to \mathbb{R}$ be a smooth function, $f_i = f(x_i)$ and
$$\omega(x) = \prod_{k=0}^{n} (x - x_k).$$
   (a) Write down the Lagrange interpolation polynomial $\mathcal{L}_n(f)(x)$ that satisfies
   $$\mathcal{L}_n(f)(x_i) = f_i$$
   for $i = 0, \cdots, n$. If $\mathcal{L}_n(1)(x) \equiv 1$ show that
   $$\mathcal{L}_n(f)(x) = \frac{\displaystyle\sum_{j=0}^{n} \frac{\lambda_j f_j}{(x - x_j)}}{\displaystyle\sum_{j=0}^{n} \frac{\lambda_j}{(x - x_j)}}$$
   where $\lambda_j = 1/\omega'(x_j)$ and prime denotes differentiation with respect to x. [**8**]

(b) For $x \in (a, b)$ show that

$$f(x) - \mathcal{L}_n(f)(x) = \frac{f^{(n+1)}(c)}{(n+1)!}\omega(x)$$

where c depends on x, $f^{(n+1)}$ denotes the $(n+1)$th derivative of f and

[**6**]

(c) Derive an upper bound for $|f(x) - \mathcal{L}_n(x)|$ if the points are equally spaced

$$x_0 = a, \quad h = (b-a)/n, \quad x_j = a + jh, \qquad j = 1, \cdots, n.$$

[**6**]